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# Lévy, Ornstein–Uhlenbeck, and Subordination: Spectral vs. Jump Description

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Unlike Brownian motion, which propagates diffusively and whose sample-path trajectories are continuous, non-Brownian Lévy motions propagate via jumps (flights) and their sample-path trajectories are purely discontinuous. When analyzing systems involving non-Brownian Lévy motions, the common practice is to use either spectral or fractional-calculus methods. In this manuscript we suggest an alternative analytical approach: using the Poisson-superposition jump structure of non-Brownian Lévy motions. We demonstrate this approach in two exemplary topics: (i) systems governed by Lévy-driven Ornstein–Uhlenbeck dynamics; and, (ii) systems subject to temporal Lévy subordination. We show that this approach yields answers and insights that are not attainable using spectral methods alone.

**KEY WORDS:** Non-Brownian Lévy motions; selfsimilar Lévy motions; Poisson superposition; Lévy-driven Ornstein–Uhlenbeck dynamics; temporal Lévy subordination.

## 1. INTRODUCTION

Lévy motions – performed by stochastic processes with stationary and independent increments – constitute one of the most important and fundamental family of random motions. Special examples of the Lévy family include the Brownian, Poisson, and Compound Poisson motions. Since their introduction in the 1930s,[(1–3)] Lévy motions were studied and researched extensively by both theoreticians and applied scientists. The literature on Lévy motions is vast, and their range of applications encompasses numerous fields of science and engineering. See refs. 4–12 for the

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theory of Lévy processes, and the references provided below for their applications.

For long years Brownian motion served as the dominant model-of-choice for random noise in continuous-time systems. This choice was based on solid grounds. Indeed, Brownian motion has many markedly appealing statistical features such as: (i) finite moments of all orders; (ii) continuous sample-path trajectories; and, (iii) selfsimilarity – Brownian motion is a 'fractal' stochastic process which is statistically invariant to changes of scale.[13] Moreover, powerful analytical methodologies to 'handle' Brownian motion are available, including: (i) the Fokker–Planck equation; (ii) the Feynman–Kac equation; and, (iii) the celebrated Ito calculus[19–22] (see also refs. 23–26).

Its remarkable statistical properties, on the one hand, and its remarkable amenability to mathematical analysis, on the other, have led Brownian motion to become *the* model of continuous-time random motion and noise. However, Brownian motion is just a single example of the Lévy family. Furthermore, it is a very special and miss-representing member of this family. We elaborate;

A main feature distinguishing Brownian motion from all other Lévy motions is the continuity of its sample-path trajectories. Amongst the Lévy family, the Brownian 'member' is the only motion with continuous sample-paths. All other motions have purely discontinuous sample-path trajectories – i.e., they are pure-jump processes. Hence, Brownian motion is the only case where the Lévy motion's propagation is conducted continuously via diffusion. In all non-Brownian cases the Lévy motion's propagation is conducted discontinuously and discretely via jumps.

Another major distinction between Brownian motion and non-Brownian Lévy motions is intimately related to the issue of selfsimilarity. As described above, a key property of Brownian motion is its scale-invariance, or selfsimilarity. However, this property is not unique to Brownian motion. The Lévy family includes an entire subset of selfsimilar motions, of which Brownian motion is only a singe example. What is unique about Brownian motion is that it is the only selfsimilar Lévy motion possessing finite variance – all other selfsimilar Lévy motions have infinite variance.

In recent years Lévy motions have drawn much attention and research.[27–39] On the one hand, numerous examples and evidence of non-Brownian noises have been discovered and documented in many 'real-world' complex systems. In fact, statistics of the Lévy type turned out to be a ubiquitous phenomena empirically observed in various areas including: physics (anomalous diffusion, turbulent flows, non-linear Hamiltonian dynamics[30,35]), biology (heartbeats,[40] firing of neural networks[41]) seismology (recordings of seismic activity[42]), electrical engineering (signal processing[43–45]), and economics (financial time series[46–48]). On the other hand, the ruling paradigm of Brownian-modeling of noise in continuous-time stochastic systems began to give way to the examination and incorporation of models driven by non-Brownian Lévy motions and noises.

When analyzing Lévy motions, the common approach is to use spectral methods: transform to Fourier or Laplace space and thus 'translate' probabilistic problems to analytic problems. An alternative approach (and, in various perspectives – an equivalent approach) is to use fractional calculus generalizations of the diffusion and Fokker–Planck equations for non-Brownian processes characterized by super-diffusive behavior.[38] These approaches have various analytical advantages. However, transforming to Fourier/Laplace space often leaves us with implicit, rather than explicit, answers to the probabilistic problems we started from. Fractional calculus methods, on the other hand, are applicable only to selfsimilar Lévy motions.

Surprisingly, the main underlying structure of non-Brownian Lévy motions – namely, their Poisson-superposition jump structure – is less commonly used. Examples exploiting this underlying structure include: in ref. 50 where Lévy-driven Langevin systems are studied and 'reverse engineered'; in ref. 51 where the extreme jumps of one-sided Lévy motions are explored; and, in ref. 52 where the first passage problem for one-sided Lévy motions is investigated.

The aim of this manuscript is to point out the 'jump-perspective' of non-Brownian Lévy motions, and to demonstrate the use of their underlying Poisson-superposition jump structure. We believe that keeping a dual analytic-oriented and jump-oriented perspectives when analyzing a system involving non-Brownian Lévy motions is indispensable and necessary in order to provide a comprehensive understanding of the system.

In order to demonstrate the use of the Poisson-superposition jump structure of non-Brownian Lévy motions we chose two exemplary topics: Lévy-driven Ornstein–Uhlenbeck dynamics,[49,50] and temporal Lévy subordination.[54,55] The Ornstein–Uhlenbeck dynamics model systems which, simultaneously, are: (i) subject to a restoring field generating a quadratic potential; and, (ii) perturbed by a continuous-time random noise. Temporal Subordination is the 'pacing mechanism' driving systems which tick according to internal subjective clocks, rather than according to the 'universal' objective clock. Subordination enables the generation of anomalous sub-diffusive and super-diffusive motions from regular diffusive motions such as Random Walks. We proceed as follows:

In Section 2 we review the notions of Lévy motions and Poisson superpositions. Section 3 is devoted to Lévy-driven Ornstein–Uhlenbeck dynamics, and Section 4 is devoted to Lévy subordination. The exposition is entirely self-contained, and examples are incorporated throughout the manuscript.

*A note about notations*: throughout the manuscript $\mathbf{P}(\cdot)$ = Probability and $\mathbf{E}[\cdot]$ = Expectation.

## 2. LÉVY MOTIONS

Lévy motions are stochastic processes with independent and stationary increments, which are continuous in probability. One-sided Lévy motions – also referred to as Lévy subordinators – are Lévy motions with non-negative increments (rendering their sample-path trajectories monotone non-decreasing). In this section we review these stochastic processes. For a comprehensive treatment of Lévy motions we refer the readers to refs. 7–10.

### 2.1. Spectral Representation

Lévy motions and one-sided Lévy motions are characterized by their spectral representations in Fourier/Laplace space. The Fourier representation of a Lévy motion $L = (L(t))_{t \geqslant 0}$ admits the form

$$\mathbf{E}[\exp\{i\omega L(t)\}] = \exp\{-\Psi_L(\omega) \cdot t\} \tag{1}$$

($\omega$ real), and the Laplace representation of a one-sided Lévy motion $S = (S(t))_{t \geqslant 0}$ admits the form

$$\mathbf{E}[\exp\{-\omega S(t)\}] = \exp\{-\Psi_S(\omega) \cdot t\} \tag{2}$$

($\omega \geqslant 0$). The functions $\Psi_L(\omega)$ and $\Psi_S(\omega)$ – the log-Fourier and log-Laplace transforms of $L(1)$ and $S(1)$ – fully characterize the underlying Lévy motion/One-sided Lévy motion, and are hence referred to as the *spectral characteristics* of $L$ and $S$. There is a one-to-one correspondence between the laws of Lévy motions and one-sided Lévy motions and *infinitely divisible* probability laws.[(53)]

Formulae (1)–(2) give the Fourier and Laplace transforms of the random variables $L(t)$ and $S(t)$ ($t \geqslant 0$). That is, Eqs. (1)–(2) are the transforms of the *one-dimensional* marginal distributions of the Lévy motion $L$ and the one-sided Lévy motion $S$. The special Lévy structure (namely; the stationarity and independence of the increments, and the continuity in probability) enables to extend Eqs. (1)–(2) to the *infinite-dimensional* Fourier and Laplace transform of the *entire* processes $L$ and $S$:

$$\mathbf{E}\left[\exp\left\{i\int_0^\infty \varphi(t)L(dt)\right\}\right]=\exp\left\{-\int_0^\infty \Psi_L(\varphi(t))dt\right\}, \quad (3)$$

and

$$\mathbf{E}\left[\exp\left\{-\int_0^\infty \varphi(t)S(dt)\right\}\right]=\exp\left\{-\int_0^\infty \Psi_S(\varphi(t))dt\right\}, \quad (4)$$

where $\varphi(t)$ is a 'nice' test function[3] (non-negative in the one-sided case). The test function $\varphi$ in Eqs. (3)–(4) is the infinite-dimensional Fourier/Laplace coordinate – the infinite-dimensional counterpart of the one-dimensional Fourier/Laplace coordinate $\omega$ in Eqs. (1)–(2).

## 2.2. Poisson Superpositions

A Poisson process, with jumps of size $x_0$ occurring at rate $\lambda_0$, is a Lévy motion and its spectral characteristic is

$$(1-\exp\{i\omega x_0\})\lambda_0.$$

A superposition of $N$ independent Poisson processes – labeled $n = 1, 2, \ldots, N$, and where the $n$th Poisson process has jumps of size $x_n$ occurring at rate $\lambda_n$ – is also a Lévy motion and its spectral characteristic is given by

$$\sum_{n=1}^{N}(1-\exp\{i\omega x_n\})\lambda_n. \quad (5)$$

Passing from Eq. (5) to a *continuum limit* – where jumps of size $x$ occur at rate $\lambda(x)dx$ – we arrive at the limiting Fourier spectral characteristic

$$\int_{-\infty}^{\infty}(1-\exp\{i\omega x\})\lambda(x)dx \quad (6)$$

($\omega$ real). If the jumps take only positive values then the counterpart of Eq. (6) is the limiting Laplace spectral characteristic

[3]That is, such that for which the integral on right hand side of (3)–(4) is well defined and convergent.

$$\int_0^\infty (1-\exp\{-\omega x\})\lambda(x)dx \tag{7}$$

($\omega \geqslant 0$). (The well-posedness of Eqs. (6)–(7) pending on the convergence of the respective integrals.)

We shall say that a real/positive-valued random variable is a *Poisson superposition* if its log-Fourier/Laplace transform is of the form of Eqs. (6)/(7), and we shall say that a Lévy motion/One-sided Lévy motion is a *Poisson superposition* if its spectral characteristic is of the form of Eqs. (6)/(7).

The rigorous formalism of Eqs. (6)–(7) is provided by the celebrated *Lévy–Khinchin theorem*. This theorem asserts that every Lévy motion can be decomposed into two *independent* stochastic parts: (i) a *continuous* part, which is a *Brownian motion* (recall that the sample-path trajectories of Brownian motion are continuous); and, (ii) a *pure-jump* part, which is a *Poisson superposition* (with a somewhat more general Fourier characteristic form than the one given in Eq. (6)).

The rate function $\lambda(x)$ appearing in Eqs. (6)–(7) will henceforth be referred to as the superposition's *Lévy–Khinchin density*. The Lévy-Khinchin density might have infinite total mass: $\int_{-\infty}^{\infty} \lambda(x)dx \leqslant \infty$. This is not due to divergence at $|x| \to \infty$ but, rather, due to a possible divergence at $|x| \to 0$. Intuitively speaking, large jumps can occur only rarely, but tiny jumps may occur very frequently. The Lévy–Khinchin density has finite total mass – i.e., $\int_{-\infty}^{\infty} \lambda(x)dx < \infty$ – if and only if the resulting motion is a *Compound Poisson* process.

## 2.3. Symmetric and One-sided Poisson Superpositions

In this paper we shall focus on two special classes of Poisson superpositions – *symmetric* and *one-sided*.

In the *symmetric* case the Lévy–Khinchin density $\lambda(x)$ is symmetric and the resulting Poisson superposition is a symmetric Lévy motion $L$ with spectral (Fourier) characteristic

$$\Psi_L(\omega) = 2\int_0^\infty (1-\cos\{\omega x\})\lambda(x)dx. \tag{8}$$

Equation (8) follows straightforwardly from Eq. (6). Due to the symmetry it is enough to specify $\Psi_L(\omega)$ for $\omega \geqslant 0$. The Lévy–Khinchin formula asserts that the admissible jump densities are such that satisfy the integrability condition

$$\int_0^\infty \min\{x^2, 1\}\lambda(x)dx < \infty \tag{9}$$

(otherwise the integral in Eq. (8) would fail to converge).

In the *one-sided* case the Lévy–Khinchin density $\lambda(x)$ vanishes on $(-\infty, 0)$ and the resulting Poisson superposition is a one-sided Lévy motion $S$ with spectral (Laplace) characteristic

$$\Psi_S(\omega) = \int_0^\infty (1 - \exp\{-\omega x\})\lambda(x)dx \tag{10}$$

($\omega \geqslant 0$). The Lévy–Khinchin formula asserts that the admissible jump densities are such that satisfy the integrability condition

$$\int_0^\infty \min\{x, 1\}\lambda(x)dx < \infty \tag{11}$$

(otherwise the integral in Eq. (10) would fail to converge).

It is both natural and useful to introduce the 'cumulative distribution function' of the Lévy–Khinchin density $\lambda(x)$. However – since the integral of $\lambda(x)$ might diverge at the origin – we need to define the 'cumulative distribution function' by integrating from $x$ to infinity, rather than from zero to $x$. This leads us to the definition of the *Lévy–Khinchin tail* function $\Lambda(x)$, $x > 0$, given by:

$$\Lambda(x) = \int_x^\infty \lambda(u)du \tag{12}$$

The meaning of the Lévy–Khinchin tail is straightforward: $\Lambda(x)$ is the *rate* at which jumps of size *greater* than $x$ occur. Formula (12) defines only the *right* Lévy–Khinchin tail. However, this is sufficient for our purposes since: (i) in the symmetric case the right and left Lévy–Khinchin tails are identical; and (ii) in the one-sided case the left Lévy–Khinchin tail vanishes.

The connection between the spectral characteristics $\Psi_L(\omega)$ and $\Psi_S(\omega)$ and the Lévy–Khinchin tail $\Lambda(x)$ is given, respectively, by

$$\frac{\Psi_L(\omega)}{\omega} = 2\int_0^\infty \sin\{\omega x\}\Lambda(x)dx \tag{13}$$

in the symmetric case, and by

$$\frac{\Psi_S(\omega)}{\omega} = \int_0^\infty \exp\{-\omega x\}\Lambda(x)dx \tag{14}$$

in the one-sided case. The derivation of Eqs. (13) and (14) follows, respectively, from Eqs. (8) and (10) using integration by parts together with the definition of the Lévy–Khinchin tail $\Lambda(x)$ (Eq. (12)).

## 2.4. Examples

To illustrate, we give below a collection of examples of symmetric and one-sided Lévy motions which are Poisson superpositions. We begin with the one-sided examples;

### *2.4.1. One-sided Poisson superpositions* $(x > 0)$

1. Compound Poisson process with Gamma-distributed jumps $(a, p > 0)$:

$$\lambda(x) = \frac{a^p}{\Gamma(p)} \exp\{-ax\} x^{p-1},$$
$$\Psi_S(\omega) = 1 - \left(\frac{a}{a+\omega}\right)^p.$$

In the special case where $p = 1$ the jumps are Exponentially distributed and $\Psi_S(\omega) = \omega/(a+\omega)$.

2. The Gamma process – one-sided Lévy motion with Gamma-distributed increments $(a > 0)$:

$$\lambda(x) = \frac{\exp\{-ax\}}{x},$$
$$\Psi_S(\omega) = \ln\left(1 + \frac{\omega}{a}\right).$$

In this case the Lévy–Khinchin density has infinite total mass, but nevertheless the motion's increments have finite moments of all orders.

3. Selfsimilar motions $(0 < \alpha < 1)$:

$$\lambda(x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}},$$
$$\Psi_S(\omega) = \omega^\alpha.$$

These processes are the only one-sided Lévy motions which are invariant under changes of scale – i.e., they are statistically selfsimilar,[13] or 'fractal'. In this case the Lévy–Khinchin density has infinite total mass, and the increments have *no* finite moments (even the mean diverges!). We shall henceforth refer to a one-sided Lévy motion with spectral characteristic $\Psi_S(\omega) = a\omega^\alpha$ as $\alpha$-*selfsimilar with amplitude* $a$.

4. Selfsimilar one-sided Lévy motions can be modified by the incorporation of an exponential cutoff, yielding ($a > 0$, $0 < \alpha < 1$):

$$\lambda(x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{\exp\{-ax\}}{x^{1+\alpha}},$$
$$\Psi_S(\omega) = (a+\omega)^\alpha - a^\alpha.$$

This cutoff retains the infinite total mass of the Lévy–Khinchin density, but renders the increments with finite moments of all orders.

5. The last example is a linear combination of examples #1 and #4 above, yielding ($a > 0$, $0 < \alpha < 1$):

$$\lambda(x) = \frac{1}{\Gamma(\alpha)} \frac{\exp\{-ax\}}{x^{1-\alpha}} \left( a + \frac{1-\alpha}{x} \right),$$
$$\Psi_S(\omega) = \frac{\omega}{(a+\omega)^\alpha}.$$

Again; the Lévy–Khinchin density has infinite total mass, but the increments have finite moments of all orders.

### *2.4.2. Symmetric Poisson superpositions* ($x \neq 0$)

1. Compound Poisson process with Laplace-distributed jumps ($a > 0$):

$$\lambda(x) = \frac{a}{2} \exp\{-a|x|\},$$
$$\Psi_L(\omega) = \frac{\omega^2}{a^2 + \omega^2}.$$

This is the two-sided counterpart of the Compound Poisson process with exponentially distributed jumps.

2. The two-sided Gamma process ($a > 0$):

$$\lambda(x) = \frac{\exp\{-a|x|\}}{|x|},$$
$$\Psi_L(\omega) = \ln\left(1 + \frac{\omega^2}{a^2}\right).$$

This is the two-sided counterpart of the (one-sided) Gamma process.

3. Selfsimilar motions ($0<\beta<2$):

$$\lambda(x)=\frac{\beta}{2I_\beta}\frac{1}{|x|^{1+\beta}}, \tag{15}$$
$$\Psi_L(\omega)=|\omega|^\beta,$$

where $I_\beta=\int_0^\infty \sin\{u\}u^{-\beta}du$. These processes are the only symmetric pure-jump Lévy motions which are invariant under changes of scale – i.e., they are statistically selfsimilar,[(13)] or 'fractal'. We shall henceforth refer to a Lévy motion with spectral characteristic $\Psi_L(\omega)=b|\omega|^\beta$ as *$\beta$-selfsimilar with amplitude $b$*. The exponent value $\beta=2$ is unattainable by Poisson superpositions. Rather, $\beta=2$ corresponds to Brownian motion: $\Psi_L(\omega)=\omega^2/2$ if and only if $L$ is a standard Brownian motion.

4. A special example of a selfsimilar motion is the Cauchy process ($\beta=1$):

$$\lambda(x)=\frac{1}{\pi}\frac{1}{|x|^2},$$
$$\Psi_L(\omega)=|\omega|.$$

5. The Cauchy process can be modified by the incorporation of an exponential cutoff, yielding ($a>0$):

$$\lambda(x)=\frac{\exp\{-a|x|\}}{|x|^2},$$
$$\Psi_L(\omega)=2|\omega|\arctan\left(\frac{|\omega|}{a}\right)-a\ln\left(1+\frac{\omega^2}{a^2}\right).$$

## 2.5. Conclusion

The Lévy–Khinchin theorem asserts that non-Brownian Lévy motions are Poisson superpositions: pure-jump processes given by the continuum superposition of Poisson processes. The common practice, when handling Lévy motions, is to employ spectral analysis and characterize the motions via their Fourier/Laplace representations. This approach is implicit, and gives no insight on the motions' underlying jump structure (in the non-Brownian case).

This jump structure, however, is given explicitly by the Lévy–Khinchin density. Indeed, the Lévy–Khinchin density specifies the '*jump frequency*' of non-Brownian Lévy motions – the Poissonian rates according to which the motions' jumps occur.

The two representations – the spectral Fourier/Laplace transform on the one hand, and the Lévy–Khinchin density on the other hand – equally characterize Lévy motions and are equally 'legitimate'. However, while the first representation is the instinctively used one, it is implicit and does not enable any intuitive way of 'picturing' the Lévy motion at hand. The second representation, on the contrary, is a truly direct and physical one – yielding the motions' exact jump frequencies.

Imagine, for example, that the one-sided Lévy motion $S$ is the cumulative claim process of an insurance company. Namely, $S(t)$ denotes the aggregate of all claims declared during the time period $[0, t]$ (i.e., the amount of money the insurance company had to pay off during this time interval). The jumps of the motion $S$ are, of course, the sizes of the incoming individual claims. Now, think of the risk managers of the insurance company: are they interested in the Laplace transform of $S$? or, are they more interested in knowing the *frequencies* according which claims of different sizes arrive?

The *tangible* information is given by the 'jump frequency' – the Lévy–Khinchin density, and not by the spectral Fourier/Laplace characteristics. Our automatic tendency to resort to spectral analysis is an acquired habit. Albeit, a very justified one: time and again does the transformation from 'physical domain' to 'spectral domain' make systems and processes amenable to mathematical analysis. However, in the case of non-Brownian Lévy motions working within the 'physical domain' *is* possible: the use of the Lévy–Khinchin density enables *direct* processing of information regarding the motions' underlying jump structure.

In the sequel we shall combine together both the spectral and jump approaches in order to analyze two exemplary topics: (i) Lévy-driven Ornstein–Uhlenbeck dynamics; and, (ii) temporal Lévy subordination.

## 3. ORNSTEIN–UHLENBECK DYNAMICS

Lévy motions serve as good approximations for stochastic process arising in complex systems and exhibiting wild types of randomness – rather than the 'mild' Gaussian type of randomness. Lévy motion models have been extensively studied and successfully employed in various fields, examples including: transport in fluid dynamics;[14] plasma physics;[15] motion patterns of biological species;[16–18] and, stock-market dynamics.[46]

Less explored – in the context of 'Lévy randomness' – has been the problem of motion in the presence of an external field. The physical cornerstone model of such motion is given by the *Ornstein–Uhlenbeck* *dynamics* – describing the motion of (diffusive) particles trapped in a restoring field generated by a *quadratic potential*. See, for example, ref. 49 and references therein.

In this section we study the behavior and dynamics of Lévy motion – rather than the regular, diffusive, Brownian motion – in the presence of a quadratic potential well. Specifically, we consider the stochastic process $Y=(Y(t))_{t\geqslant 0}$ generated by the following *Lévy-driven Ornstein–Uhlenbeck dynamics*:

$$\dot{Y}(t)=-\kappa Y(t)+\dot{X}(t), \tag{16}$$

where:

- $\kappa>0$ is the amplitude of the 'retrieving force' exerted by the quadratic potential; and,
- $\dot{X}$ is a driving 'Lévy noise' – the derivative of a Lévy motion/One-sided Lévy motion $X=(X(t))_{t\geqslant 0}$ with spectral characteristic $\Psi_X(\omega)$.

### 3.1. Spectral Analysis

Integration by parts of Eq. (16) yields

$$Y(t)=\exp\{-\kappa t\}Y(0)+\int_0^t \exp\{-\kappa(t-s)\}X(ds). \tag{17}$$

Let $\Psi_{Y(t)}(\omega)$ denote the log-Fourier/Laplace transform of $Y(t)$, $t\geqslant 0$. If the initial condition $Y(0)$ is independent of the Lévy driver $X$ then transforming Eq. (17) to Fourier/Laplace space and using Eqs. (3)–(4) yields

$$\Psi_{Y(t)}(\omega)=\Psi_{Y(0)}\left(\omega\exp\{-\kappa t\}\right)+\int_0^t \Psi_X(\omega\exp\{-\kappa(t-s)\})ds.$$

Or, equivalently;

$$\Psi_{Y(t)}(\omega)=\Psi_{Y(0)}\left(\omega\exp\{-\kappa t\}\right)+\frac{1}{\kappa}\int_{|\omega|\exp\{-\kappa t\}}^{|\omega|}\frac{\Psi_X(\mathrm{sign}(\omega)u)}{u}du.$$

Finally, taking $t\to\infty$, we obtain that the process $Y$ converges to a stationary limit $Y(\infty)=\lim_{t\to\infty}Y(t)$ with log-Fourier/Laplace transform

$$\Psi_Y(\omega)=\frac{1}{\kappa}\int_0^{|\omega|}\frac{\Psi_X(\mathrm{sign}(\omega)u)}{u}du \tag{18}$$

($\omega$ real in case $X$ is a Lévy motion, and $\omega \geqslant 0$ in case $X$ is a one-sided Lévy motion).

### 3.2. The Ornstein–Uhlenbeck Map

Based on the stochastic dynamics of Eq. (16) and on the spectral analysis of subsection 3.1, we introduce the transformation

$$X \xrightarrow{\mathbf{T}} Y, \tag{19}$$

mapping the *distribution* of the 'input' Lévy driver $X$ (namely, the law of $X(1)$) to the *stationary distribution* of the 'output' Ornstein–Uhlenbeck process $Y$ (namely, the law of $Y(\infty)$). We coin the transformation $\mathbf{T}$ the *Ornstein–Uhlenbeck map*.

We henceforth assume that the Lévy driver $X$ is either a symmetric or a one-sided Poisson superposition, and (with no loss of generality) that $\kappa = 1$. Thus, Eq. (18) implies that the *spectral representation* of the Ornstein–Uhlenbeck map $\mathbf{T}$ is

$$\Psi_Y(\omega) = (\mathbf{T}\Psi_X)(\omega) = \int_0^{\omega} \frac{\Psi_X(u)}{u} du, \tag{20}$$

where $\Psi_X(\omega)$ and $\Psi_Y(\omega)$ are, respectively, the log-Fourier/Laplace transforms of $X(1)$ and $Y(\infty)$, and where $\omega \geqslant 0$.

Now; if $X$ is a Poisson superposition, what can we say about $Y$? Is it a Poisson superposition as well? And if it is – then what is its Lévy–Khinchin density? These questions are left unanswered by Eq. (20) – which provides us only with the *implicit* spectral representation of $\mathbf{T}$. Rather, we wish to understand the transformation of the *jump structure* caused by the Ornstein–Uhlenbeck map $\mathbf{T}$. The answer to these questions is given by the following proposition;

**Proposition 1.** Assume that the input Lévy driver $X$ is a symmetric/one-sided Poisson superposition with Lévy–Khinchin density $\lambda_X(x)$ and Lévy–Khinchin tail $\Lambda_X(x)$ $(x > 0)$. Then, the stationary distribution of the output Ornstein–Uhlenbeck process $Y$ is a symmetric/one-sided Poisson superposition, and its Lévy-Khinchin density and Lévy-Khinchin tail are given, respectively, by $(y > 0)$:

$$\lambda_Y(y) = (\mathbf{T}\lambda_X)(y) := \frac{1}{y} \int_y^{\infty} \lambda_X(x) dx, \tag{21}$$

and

$$\Lambda_Y(y) = (\mathbf{T}\Lambda_X)(y) := \int_y^\infty \frac{\Lambda_X(x)}{x} dx. \tag{22}$$

In other words, the representation of the Ornstein–Uhlenbeck map **T** in 'jump space' (rather than in Fourier/Laplace space) is given by either of Eqs. (21) and (22): Eq. (21) representing the transformation of the *Lévy–Khinchin density*; and, Eq. (22) representing the transformation of the *Lévy–Khinchin tail*. The proof of proposition 1 is given in Appendix A.

Three remarks and one corollary are in place:

(a) The connection between the input's Lévy–Khinchin tail $\Lambda_X(x)$ and the output's Lévy–Khinchin density $\lambda_Y(y)$ is given by

$$\lambda_Y(y) = \frac{1}{y}\Lambda_X(y). \tag{23}$$

(b) Observe the structural resemblance between the jump-tail representation of Eq. (22) and the spectral representation of Eq. (20).

(c) Note that in all its three representations (Eqs. (20 )–(22)) the Ornstein–Uhlenbeck map turns out to be an *integral operator* – smoothening out the input.

(d) Since the harmonic function $1/x$ is not integrable at the origin, Eq. (23) implies that – regardless of the input $X$ – we always have $\lim_{y\to 0}\Lambda_Y(y) = \infty$. This, in turn, yields the following corollary:

*The output of the Ornstein–Uhlenbeck map* **T** *is never Compound Poisson.* (24)

The overall infinite Poissonian jump rate is, in a sense, the 'price-payed' for the smoothening effect of the Ornstein-Uhlenbeck map.

## 3.3. Examples

Let us give a few examples of the Ornstein–Uhlenbeck map **T** 'in action';

*The one-sided case* (The Examples we refer to below are the one-sided Poisson superposition examples of subsection 2.4)

1. The Compound Poisson process with exponentially distributed jumps (Example #1) maps to the Gamma process (Example #2) ($a > 0$):

$$\lambda_X(x) = a \exp\{-ax\} \Rightarrow \lambda_Y(y) = \frac{\exp\{-ay\}}{y}.$$

2. The $\alpha$-selfsimilar one-sided Lévy motions (Example #3) map to themselves ($0 < \alpha < 1$):

$$\lambda_X(x) = \frac{\alpha}{x^{1+\alpha}} \Rightarrow \lambda_Y(y) = \frac{1}{y^{1+\alpha}}.$$

3. The one-sided Lévy motion of Example #5 maps to the one-sided Lévy motion of Example #4 (selfsimilar modified by an exponential cut-off) ($a > 0,\ 0 < \alpha < 1$):

$$\lambda_X(x) = \frac{\exp\{-ax\}}{x^{\alpha}} \left(a + \frac{\alpha}{x^{\alpha}}\right) \Rightarrow \lambda_Y(y) = \frac{\exp\{-ay\}}{y^{1+\alpha}}.$$

*The symmetric case* (The Examples we refer to below are the symmetric Poisson superposition examples of subsection 2.4)

1. The Compound Poisson process with Laplace-distributed jumps (Example #1) maps to the two-sided Gamma process (Example #2) ($a > 0$):

$$\lambda_X(x) = a \exp\{-a|x|\} \Rightarrow \lambda_Y(y) = \frac{\exp\{-a|y|\}}{|y|}.$$

2. The $\beta$-selfsimilar one-sided Lévy motions (Example #3) map to themselves ($0 < \beta < 2$):

$$\lambda_X(x) = \frac{\beta}{|x|^{1+\beta}} \Rightarrow \lambda_Y(y) = \frac{1}{|y|^{1+\beta}}.$$

3. The linear combination of the two-sided Gamma process (Example #2) and the Cauchy process with exponential cutoff (Example #5) maps to a Cauchy process with exponential cutoff:

$$\lambda_X(x) = \frac{\exp\{-a|x|\}}{|x|} \left(a + \frac{1}{|x|}\right) \Rightarrow \lambda_Y(y) = \frac{\exp\{-a|y|\}}{|y|^2}.$$

### 3.4. Inversion

A question naturally arising in the context of the Ornstein–Uhlenbeck map is that of '*reverse-engineering*': what input $X$ would yield a desired output $Y$? In this subsection we explore the *inverse transformation* $\mathbf{T}^{-1}$ of the Ornstein–Uhlenbeck map $\mathbf{T}$:

$$X \overset{\mathbf{T}^{-1}}{\longleftarrow} Y,$$

Let us first analyze the *image* of the Ornstein–Uhlenbeck map $\mathbf{T}$. From Eq. (21) it is evident that the function $\lambda_Y(y)$ is always of the functional form $g(y)/y$ where $g(y)$ is a smooth and monotone decreasing function with $\lim_{y\to\infty} g(y) = 0$. Furthermore, the function $g(y)$ must comply with the integrability conditions of Eqs. (9) and (11). Namely, $g(y)$ must satisfy

$$\int_0^\infty \min\left\{y, \frac{1}{y}\right\} g(y)dy < \infty$$

in the symmetric case, and

$$\int_0^\infty \min\left\{1, \frac{1}{y}\right\} g(y)dy < \infty$$

in the one sided case.

Having set an admissible function $g(y)$ (satisfying the above mentioned criteria) the '*reverse-engineering recipe*' is given by the following simple rule:

$$\lambda_X(x) = -g'(x) \Rightarrow \lambda_Y(y) = \frac{g(y)}{y}. \tag{25}$$

That is, the input Lévy–Khinchin density yielding an output with pre-specified Lévy-Khinchin density $\lambda_Y(y) = g(y)/y$ is $\lambda_X(x) = -g'(x)$. Equation (25) is obtained by differentiation of Eq. (21). We give three examples;

1. $g(y) = \exp\{-y^p\}y^{-q}$:

$$\lambda_X(x) = \frac{\exp\{-x^p\}}{x^{1+q}}\left(px^p + q\right) \Rightarrow \lambda_Y(y) = \frac{\exp\{-y^p\}}{y^{1+q}},$$

where $p > 0$ and: $0 \leqslant q < 1$ in the one-sided case; $0 \leqslant q < 2$ in the symmetric case.

2. $g(y) = (c+y)^{-p} y^{-q}$:

$$\lambda_X(x) = \frac{(p+q)x + cq}{(c+x)^{1+p} x^{1+q}} \Rightarrow \lambda_Y(y) = \frac{1}{(c+y)^p y^{1+q}},$$

where $c, p > 0$ and: $0 \leqslant q < 1$ in the one-sided case; $0 \leqslant q < 2$ in the symmetric case.

3. $g(y) = -\frac{1}{c} \ln(1 - \exp\{-cy\})$:

$$\lambda_X(x) = \frac{\exp\{-cx\}}{1 - \exp\{-cx\}} \Rightarrow \lambda_Y(y) = \frac{1}{cy} \ln\left(\frac{1}{1 - \exp\{-cy\}}\right),$$

where $c > 0$. (The function $z = g(y)$ is obtained from the curve $\exp\{-cy\} + \exp\{-cz\} = 1$.)

We point out that in all the three examples *no* closed-form analytic expression for the spectral characteristics of neither the input $X$ nor the output $Y$ exist. Nevertheless, the jump densities are easily and explicitly computable.

To conclude this subsection, we note that the inversions of the spectral representation (20) and the inversion of the jump-tail representation (22) are given, respectively, by

$$\Psi_X(\omega) = \omega \Psi_Y'(\omega),$$

and

$$\Lambda_X(x) = -x \Lambda_Y'(x).$$

## 3.5. Eigen-distributions

In subsection 3.3 we have seen that the Ornstein–Uhlenbeck transformation **T** maps selfsimilar Poisson superpositions to themselves. In other words, selfsimilar Poisson superpositions are *eigen-distributions* of the Ornstein–Uhlenbeck map **T**. Are they the *only* eigen-distributions of **T**? This is the issue of this subsection.

Consider the eigenvalue problem for the Ornstein–Uhlenbeck map **T**. In the spectral representation it is given by

$$\int_0^\omega \frac{\Psi(u)}{u} du = \eta \Psi(\omega) \tag{26}$$

($\omega \geqslant 0$), and in the jump-tail representation it is given by

$$\int_y^\infty \frac{\Lambda(x)}{x} dx = \mu \Lambda(y) \tag{27}$$

($y > 0$).

The solutions of the eigenvalue problems (26)–(27) are given, respectively, by

$$\Psi(\omega) = \Psi(1)\omega^{1/\eta}, \tag{28}$$

and

$$\Lambda(x) = \Lambda(1)x^{-1/\mu}. \tag{29}$$

Both Eqs. (28) and (29) imply that the eigen-distributions of the Ornstein–Uhlenbeck map **T** are the selfsimilar Poisson superpositions. In the symmetric case the range of the map's eigenvalues is $(1/2, \infty)$, and in the one-sided case the range is $(1, \infty)$. Note that the eigenvalues $\mu$ and $\eta$ turned out to be the *Hurst exponents* of the selfsimilar Lévy motions and one-sided Lévy motions (namely; $1/\beta$ in the symmetric case, and $1/\alpha$ in the one-sided case). Furthermore, we have obtained the following Ornstein–Uhlenbeck *characterization* of selfsimilar Poisson superpositions:

**Proposition 2.** A one-sided/symmetric Poisson superposition is selfsimilar if and only if its law is an eigen-distribution of the Ornstein–Uhlenbeck map **T**.

## 4. TEMPORAL SUBORDINATION

*Temporal Subordination* arises naturally in systems whose subjective 'operational time' is different from the objective 'physical time'.[53] That is, systems which tick according to an internal – often stochastic and irregular – 'subjective clock', rather than pace according to the universal 'objective clock' (whose time flow is deterministic and linear). Furthermore, Temporal Subordination is a most effective mean of introducing anomalies into diffusive processes *without* distorting or changing their underlying transport mechanisms. In particular, subordination elegantly produces both subdiffusive and super-diffusive motions from regular diffusive motions such as Random Walks and Brownian motion. See, for example, refs. 54, 55.

In this section we consider the Temporal Subordination of general symmetric Lévy motions. Specifically, we assume fixed a symmetric Lévy motion $L=(L(t))_{t\geqslant 0}$. Then, given a one-sided Lévy motion $X=(X(t))_{t\geqslant 0}$, we introduce the process $Y=(Y(t))_{t\geqslant 0}$ generated by the following subordination of the motion $L$ by the motion $X$:

$$Y(t)=L(X(t)). \tag{30}$$

## 4.1. The Subordination Map

The resulting $Y$ process defined in Eq. (30) is a symmetric Lévy motion. Hence $L$ induces a transformation

$$X \overset{\mathbf{S}}{\longmapsto} Y, \tag{31}$$

mapping *one-sided* Lévy motions to *symmetric* Lévy motions. We coin the transformation $\mathbf{S}$ the *Subordination map*.

Let $\Psi_L$, $\Psi_X$, and $\Psi_Y$ denote, respectively, the spectral characteristics of $L$, $X$, and $Y$. Straightforward conditioning implies that:

$$\Psi_Y(\omega)=(\mathbf{S}\Psi_X)(\omega)=\Psi_X(\Psi_L(\omega)). \tag{32}$$

In other words, Eq. (32) gives the spectral representation of the subordination map $\mathbf{S}$. Note that the stochastic sample-path composition of processes $Y=L\circ X$ transforms – when passing to spectral representation – in a '*reverse order*' to the functional composition of their spectral characteristics $\Psi_Y=\Psi_X\circ\Psi_L$.

Now; if $X$ is a Poisson superposition, what can we say about $Y$? Is it a Poisson superposition as well? And if it is – then what is its Lévy–Khinchin density? These questions are left unanswered by Eq. (32), which provides us only with the *implicit* spectral representation of $\mathbf{S}$. Rather – as in Section 3 – we wish to understand the transformation of the *jump structure* caused by the subordination map $\mathbf{S}$. The answer to these questions is given by the following proposition;

**Proposition 3.** Let $f(t;\cdot)$ denote the probability density function of $L(t)$, $t\geqslant 0$. Assume that the input one-sided Lévy motion $X$ is a one-sided Poisson superposition with Lévy–Khinchin density $\lambda_X(x)$ $(x>0)$. Then, the output Lévy motion $Y$ is a symmetric Poisson superposition, and its Lévy–Khinchin density is given by $(y>0)$:

$$\lambda_Y(y)=\int_0^\infty f(x;y)\lambda_X(x)dx. \tag{33}$$

The proof of proposition 3 is given in Appendix A. The probability density function $f$ of the Lévy motion $L$ serves as an *integration kernel* in Eq. (33). Note that the integration on the right hand side of Eq. (33) is along the *temporal coordinate* of $f$ (and *not* along $f$'s spatial coordinate). The connection between the spectral characteristic $\Psi_L$ of the motion $L$ and its probability density function $f$ is given inverse Fourier transform of Eq. (1):

$$f(x;y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-i\omega y\}\exp\{-\Psi_L(\omega)x\}d\omega \tag{34}$$

Integrating both sides of Eq. (33), while taking into account that $\int_0^\infty f(x;y)dy = 1$ (for all $x$), yields the equality $\Lambda_Y(0) = \Lambda_X(0)$. That is, the overall 'jump rate' of the input $X$ and the output $Y$ are equal. This, in turn, yields the following corollary:

$$\begin{array}{c} \textit{The output of the Subordination map } \mathbf{S} \textit{ is} \\ \textit{Compound Poisson if and only the input is such.} \end{array} \tag{35}$$

Note that the corollary stated in Eq. (35) is the 'subordination counterpart' of the corollary stated in Eq. (24) regarding the Ornstein–Uhlenbeck map **T**.

### 4.2. The Brownian, Cauchy, and Selfsimilar Cases: Probabilistic Representations

In two special cases – when $L$ is a Brownian motion ($\Psi_L(\omega) = \omega^2/2$), and when $L$ is a Cauchy motion ($\Psi_L(\omega) = |\omega|$) – the explicit form of the density kernel $f(x;y)$ is known. In the general selfsimilar case ($\Psi_L(\omega) = |\omega|^\beta$, $\beta \neq 1, 2$) the explicit form of $f(x;y)$ is unknown, but $f(x;y)$ nevertheless satisfies a scaling property. In all these three cases, as we shall demonstrate in this subsection, Eq. (33) admits *probabilistic representations*.

Throughout this subsection we shall use the shorthand notation

$$g_X(x) = x\lambda_X(x) \quad \text{and} \quad g_Y(y) = y\lambda_Y(y). \tag{36}$$

#### *4.2.1. The Brownian Case*

When $L$ is a Brownian motion the density kernel $f(x;y)$ is given by

$$f(x;y) = \frac{1}{\sqrt{2\pi x}}\exp\left\{-\frac{y^2}{2x}\right\}. \tag{37}$$

This kernel is intimately related to the probability density function of the Lévy–Smirnoff distribution – the distribution of the *first passage times* of Brownian motion. We explain;

Let $\tau(y)$ denote the first time a Brownian motion (starting from the origin) hits the level $y$ $(y>0)$. Namely, $\tau(y):=\inf\{t\geqslant 0|L(t)=y\}$. It is well known (see, for example, ref. 26) that the probability density function of $\tau(y)$ is given by $(x>0)$:

$$f_{\tau(y)}(x)=\frac{y}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2x}\right\}\frac{1}{x^{3/2}}.$$

Now; substituting the kernel (37) into Eq. (33) and using the shorthand notation of Eq. (36) yields

$$g_Y(y)=\int_0^\infty f_{\tau(y)}(x)g_X(x)dx.$$

Hence, we have obtained a probabilistic representation of Eq. (33) based on the *first passage times* of Brownian motion:

$$g_Y(y)=\mathbf{E}[g_X(\tau(y))]. \tag{38}$$

Furthermore, it is well known that the first passage times – when viewed as a *process* $(\tau(y))_{y\geqslant 0}$ in the parameter $y$ – form a $\frac{1}{2}$-selfsimilar one-sided Lévy motion with amplitude $\sqrt{2}$. This gives yet another meaning to the representation (38). In particular, $\mathbf{E}[\exp\{-\omega\tau(y)\}]=\exp\{-\sqrt{2\omega}\cdot y\}$ $(\omega\geqslant 0)$ and hence Eq. (38) implies the following equivalence: $g_X(x)$ is the Laplace transform of the function $\phi(u)$ if and only if $g_Y(y)$ is the Laplace transform of the function $\phi(u^2/2)u$. Namely;

$$g_X(x)=\int_0^\infty \exp\{-xu\}\phi(u)du \quad\Leftrightarrow\quad g_Y(y)=\int_0^\infty \exp\{-yu\}\left(\phi\left(\frac{u^2}{2}\right)u\right)du.$$

### 4.2.2. The Cauchy Case

When $L$ is the Cauchy motion the density kernel $f(x;y)$ is given by

$$f(x;y)=\frac{x}{\pi}\frac{1}{x^2+y^2}. \tag{39}$$

Substituting the kernel (39) into Eq. (33) and using the shorthand notation of Eq. (36) yields

$$g_Y(y) = \int_0^\infty f(y; x) g_X(x) dx. \tag{40}$$

Note that the order of $x$ and $y$ in the integration kernel of Eq. (40) are reversed: $f(x; y)$ in Eq. (33) reverses to $f(y; x)$ in Eq. (40).

Equation (40), in turn, gives us a probabilistic representation of Eq. (33) based on the absolute values of the Cauchy motion $L$:

$$g_Y(y) = \frac{1}{2}\mathbf{E}\left[g_X(|L(y)|)\right]. \tag{41}$$

### *4.2.3. The General Selfsimilar Case*

When $L$ is selfsimilar then the density kernel $f(x; y)$ satisfies a scaling property. Namely; $L$ is $\beta$-selfsimilar if and only if

$$f(x; y) = f\left(1; \frac{y}{x^{1/\beta}}\right) \frac{1}{x^{1/\beta}}. \tag{42}$$

Substituting the kernel (42) into Eq. (33) and using the shorthand notation of Eq. (36) yields

$$g_Y(y) = \int_0^\infty f\left(1; \frac{y}{x^{1/\beta}}\right) \frac{y}{x^{1/\beta}} \frac{g_X(x)}{x} dx$$

which, in turn (using the change of variables $z = y/x^{1/\beta}$), gives

$$g_Y(y) = \beta \int_0^\infty f(1; z)\, g_X\left(\frac{y^\beta}{z^\beta}\right) dz.$$

Hence, we obtain the probabilistic representation

$$g_Y(y) = \frac{\beta}{2}\mathbf{E}\left[g_X\left(\frac{y^\beta}{|L(1)|^\beta}\right)\right]. \tag{43}$$

### 4.2.4. Back to the Brownian and Cauchy cases

The probabilistic representation of Eq. (43) can be used, in particular, in the Brownian ($\beta=2$) and Cauchy ($\beta=1$) cases[4]:

- The Brownian case: for $\beta=2$ Eq. (43) yields

$$g_Y(y)=\mathbf{E}\left[g_X\left(\frac{y^2}{Z^2}\right)\right], \tag{44}$$

where $Z$ is a normalized Gaussian random variable (i.e., with zero mean and unit variance). Equation (44), at first glance, does not seem to coincide with the representation (38). However – since $\tau(y)\overset{d}{=}y^2/Z^2$ – the representations (38) and (44) are indeed identical.

- The Cauchy case: for $\beta=1$ Eq. (43) yields

$$g_Y(y)=\frac{1}{2}\mathbf{E}\left[g_X\left(\frac{y}{|Z|}\right)\right], \tag{45}$$

where $Z$ is a normalized Cauchy random variable (i.e., with Fourier transform $\exp\{-|\omega|\}$). Again, the representations (45) and (41) do not seem, at first glance, to coincide. However, they *do* since $L(y)\overset{d}{=}yZ$ and since $Z\overset{d}{=}1/Z$ (i.e., the normalized Cauchy random variable $Z$ is equal, in law, to its reciprocal $1/Z$).

## 4.3. Examples

### 4.3.1. Brownian Subordinated by Selfsimilar

Let $L$ be a Brownian motion, and take $X$ to be an $\alpha$-selfsimilar one-sided Lévy motion ($0<\alpha<1$) with amplitude $2^\alpha$. Then $\Psi_L(\omega)=\omega^2/2$ and $\Psi_X(\omega)=2^\alpha\omega^\alpha$, and hence Eq. (32) yields $\Psi_Y(\omega)=|\omega|^{2\alpha}$. That is, $Y$ is $2\alpha$-selfsimilar with unit amplitude.

This example shows us that when $L$ is a Brownian motion then the subordination map $\mathbf{S}$ establishes a *one-to-one* and *onto* correspondence between selfsimilar one-sided Lévy motions and selfsimilar (non-Brownian) symmetric Lévy motions. The range $0<\alpha<1$ (of the selfsimilarity exponent of one-sided Lévy motions) is mapped to the range $0<\beta<2$ (of

[4]Below $\overset{d}{=}$ stands for equality in distribution (law).

the selfsimilarity exponent of symmetric motions) by the linear transformation $\beta = 2\alpha$.

Furthermore, using Eq. (44) we obtain a simple way of deriving a closed-form analytic expression for the following (hard!) integral[5]:

$$I_{2\alpha} := \int_0^\infty \frac{\sin\{u\}}{u^{2\alpha}} du = \frac{\sqrt{\pi}}{4^\alpha} \frac{\Gamma(1-\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)} \quad (0<\alpha<1). \tag{46}$$

This integral appears in the coefficient of the Lévy–Khinchin density of symmetric selfsimilar Lévy motions (see Eq. (15)). We shall use Eq. (46) below in order to compute the fractional moments of symmetric selfsimilar Lévy motions (see the proof of Eq. (47)). In particular, Eq. (46) implies that $\int_0^\infty (\sin\{u\}/u) du = \pi/2$ and $\int_0^\infty (\sin\{u\}/\sqrt{u}) du = \sqrt{\pi/2}$. The derivation of Eq. (46) is given in Appendix A.

#### 4.3.2. Selfsimilar Subordinated by Selfsimilar

Let $L$ be a $\beta$-selfsimilar $(0<\beta<2)$ Lévy motion with unit amplitude, and take $X$ to be an $\alpha$-selfsimilar $(0<\alpha<1)$ one-sided Lévy motion with unit amplitude. Then $\Psi_L(\omega) = |\omega|^\beta$ and $\Psi_X(\omega) = \omega^\alpha$, and hence Eq. (32) yields $\Psi_Y(\omega) = |\omega|^{\alpha\beta}$.

This example shows us that when $L$ is a $\beta$-selfsimilar Lévy motion then the subordination map **S** establishes a *one-to-one* and *onto* corre-Spondence between selfsimilar one-sided Lévy motions and the *subrange* $(0, \alpha\beta]$ of selfsimilar symmetric Lévy motions.

Furthermore, using Eq. (43) we obtain a simple way of deriving the *fractional moments* of order $0<p<\beta$ (higher-order moments diverge) of the selfsimilar Lévy motion $L$:

$$\mathbf{E}\left[|L(t)|^p\right] = \frac{2^p}{\sqrt{\pi}} \Gamma\left(\frac{1+p}{2}\right) \frac{\Gamma(1-p/\beta)}{\Gamma(1-p/2)} \cdot t^{p/\beta}. \tag{47}$$

The derivation of Eq. (47) is given in Appendix A.

#### 4.3.3. Brownian Subordinated by Gamma

Let $L$ be a Brownian motion, and take $X$ to be a Gamma motion with parameter $a$. Then $\Psi_L(\omega) = \omega^2/2$ and $\Psi_X(\omega) = \ln(1+\omega/a)$, and hence Eq. (32) yields $\Psi_Y(\omega) = \ln(1+\omega^2/2a)$.

[5]See also ref. 56.

For the Gamma motion $g_X(x)=\exp\{-ax\}$ and hence Eq. (38) implies that $g_Y(y) = \mathbf{E}[\exp\{-a\tau(y)\}]$. However, $\mathbf{E}[\exp\{-a\tau(y)\}]$ is the Laplace transform of $\tau(y)$ evaluated at the point $a$ – which, in turn, is well known to equal $\exp\{-\sqrt{2a}\cdot y\}$ (see, for example, ref. 26). Hence we obtain that $g_Y(y)=\exp\{-\sqrt{2a}\cdot y\}$ - which confirms with the two-sided Gamma example of subsection 2.4.

### *4.3.4. A Brownian Example*

Let $L$ be a Brownian motion, and take $X$ to be a one-sided Lévy motion with Lévy–Khinchin density

$$\lambda_X(x)=c_p\frac{\exp\{-1/2x\}}{x^{1+p/2}},$$

where $p>0$ and $c_p=\sqrt{\pi}/\left(2^{p/2}\Gamma\left(\frac{1+p}{2}\right)\right)$. A straightforward application of formula (33), using the Gaussian kernel (37), gives

$$\lambda_Y(y)=\left(\frac{1}{\sqrt{1+y^2}}\right)^{1+p}.$$

Note that in this example there are *no* closed-form analytic expressions for neither the spectral characteristic of $X$ nor the spectral characteristic of $Y$. Yet, computations involving the jump densities are simple and yield explicit results.

### *4.3.5. A Cauchy example*

Let $L$ be a Cauchy motion, and take $X$ to be a one-sided Lévy motion with Lévy–Khinchin density

$$\lambda_X(x)=\frac{1}{c^2+x^2},$$

where $c>0$. A straightforward application of formula (33), using the Cauchy kernel (39), gives

$$\lambda_Y(y)=2\frac{\ln(y)-\ln(c)}{y^2-c^2}.$$

As in the previous example – there are *no* closed-form analytic expressions for neither the spectral characteristic of $X$ nor the spectral characteristic of $Y$. Yet, again, computations involving the jump densities are simple and yield explicit results.

## 5. CONCLUSIONS

Brownian motion and spectral analysis are employed ubiquitously in science and engineering. Brownian motion is the predominant 'model-of-choice' used to describe noise in continuous-time systems. Fourier and Laplace transforms are the predominant 'tool-of-choice' in the analysis of time series and temporal data. Be it a quantitative problem in physics, chemistry, biology, or engineering – we instinctively use Brownian motion to model noise, if present; and, we automatically apply Fourier and Laplace analysis to unveil underlying spectral patterns, if existing.

However, noise is often so much not Brownian, and randomness is often so much not Gaussian.[57] When fluctuations are continuous and mild, then indeed Brownian is the noise. However, when fluctuations are discontinuous and wild, then the noise is not Brownian – rather, it is *Lévy*. In recent years 'Lévy randomness' began to draw much attention. However, 'Lévy randomness' is much more complicated and intricate than 'Brownian randomness', and is far less amenable to mathematical analysis.

The mathematical toolkit at our disposal, when tackling randomness of the Lévy type, is painfully small. It contains merely three instruments: spectral analysis, fractional calculus, and the Lévy–Khinchin theorem. The second instrument – fractional calculus – is in various perspectives equivalent to spectral analysis, and is applicable only in cases where the Lévy randomness is statistically selfsimilar. Thus, essentially we are left only with spectral analysis and with the Lévy–Khinchin theorem. Effectively, however, scientists use almost exclusively spectral analysis. And it should not be so!

Spectral analysis and the Lévy–Khinchin theorem both enable the exact characterization of Lévy type randomness. The spectral characteristic is an implicit one – it codes information in an intangible way. Knowing the spectral characteristic of a Lévy motion renders no clue regarding the motion's underlying structure: its *jumps*. On the other hand, Lévy–Khinchin theorem does precisely what spectral analysis does not: its *Lévy–Khinchin density* fully specifies the underlying jump structure of Lévy motions. Namely, it gives the entire '*jump frequencies*' of a Lévy motion – the exact Poissonian rates at which the motion's jumps occur.

Oddly, this most informative characteristic is seldom used by physicists. Albeit, the Lévy–Khinchin density is a very physical, tangible, and illustrative concept. We call out to physicists and scientists at large:

*When working with Lévy type randomness, always combine spectral analysis together with 'jump analysis' – it will pay out!*

To support this motto, we demonstrate the combined use of spectral analysis and 'jump analysis' in the exploration of two exemplary topics – *Ornstein–Uhlenbeck dynamics* and *Temporal Subordination*. Both these topics are cornerstones in physics: the former being a most fundamental form of dynamics; the latter being the core mechanism of systems pacing according to an internal and subjective 'operational time' which differs from the external and objective 'physical time'. In both these exemplary topics, when the input is of a Lévy type then so is the output. Thus these systems induce '*mappings of randomness*' which transform input Lévy randomness to output Lévy randomness. Studying these maps, incorporating both spectral and jump analyses, we:

- obtain closed form formulae for the transformation of the spectral characteristics and the Lévy–Khinchin densities under the Ornstein-Uhlenbeck and Subordination maps;
- prove that the output of the Ornstein–Uhlenbeck map is never Compound Poisson, whereas the output of the Subordination map is Compound Poisson if and only if its input is so; and,
- give examples of systems where spectral analysis is not implementable (i.e., neither the characteristic input nor the characteristic output are computable), whereas the 'jump analysis' easily yields explicit results.

Moreover, for Ornstein–Uhlenbeck systems:

- a closed form '*reverse-engineering*' scheme is devised, telling us what type of input Lévy randomness is required in order to yield a predesired output Lévy randomness; and,
- the eigenvalue-problem of the Ornstein–Uhlenbeck map is analyzed, and the map's set of eigen-distributions is shown to *coincide* with the class of statistically selfsimilar Lévy laws – yielding, in turn, a novel characterization of these laws.

Most important, however, is the fact that the 'jump analysis' incorporated enables us to vividly *picture* the action of these maps. It enables us to *see* how the curve of the input jump frequencies transforms to the corresponding output frequency curve. We thus conclude with re-iterating the above stated call out: *jump analysis may let you vividly see what spectral analysis may hide cloaked, use it – for it is indispensable a tool.*

## APPENDIX A.

### A.1. Proposition 1

*Proof.* The jump-tail representation (22) follows straightforwardly from the jump-density representation (21) by integration. As for the proof of Eq. (21) – we split to the symmetric and one-sided cases:

*The symmetric case*

Using formulae (20), (13) and (8) we have

$$\begin{aligned}\Psi_Y(\omega) &= \int_0^\omega \frac{\Psi_X(u)}{u} du \\ &= \int_0^\omega \left(2\int_0^\infty \sin\{yu\}\Lambda_X(y)dy\right) du \\ &= 2\int_0^\infty (1-\cos\{\omega y\})\left(\frac{\Lambda_X(y)}{y}dy\right),\end{aligned}$$

and hence $\lambda_Y(y)=\Lambda_X(y)/y$ – proving Eq. (21).

*The one-sided case*

Using formulae (20), (14) and (10) we have

$$\begin{aligned}\Psi_Y(\omega) &= \int_0^\omega \frac{\Psi_X(u)}{u} du \\ &= \int_0^\omega \left(\int_0^\infty \exp\{-yu\}\Lambda_X(y)dy\right) du \\ &= \int_0^\infty (1-\exp\{-\omega y\})\left(\frac{\Lambda_X(y)}{y}dy\right),\end{aligned}$$

and hence $\lambda_Y(y)=\Lambda_X(y)/y$ – proving Eq. (21). ∎

### A.2. Proposition 3

*Proof.* (Throughout the proof we use the shorthand notation $\Psi = \Psi_L$)

By definition

$$\Psi_X(\theta)=\int_0^\infty (1-\exp\{-\theta x\})\lambda_X(x)dx, \tag{48}$$

and

$$\Psi_Y(\omega) = \int_{-\infty}^{\infty} (1 - \exp\{i\omega y\})\lambda_Y(y)dy. \tag{49}$$

Differentiating Eq. (48) gives

$$\Psi'_X(\theta) = \int_0^{\infty} \exp\{-\theta x\}) \, [x\lambda_X(x)] \, dx. \tag{50}$$

Differentiating Eq. (49), while using the functional composition $\Psi_Y = \Psi_X \circ \Psi$ we obtain that

$$\Psi'_X \, (\Psi(\omega)) \, \Psi'(\omega) = -i \int_{-\infty}^{\infty} \exp\{i\omega y\} \, [y\lambda_Y(y)] \, dy,$$

which in turn (using Fourier inversion) implies that

$$\lambda_Y(y) = \frac{i}{2\pi y} \int_{-\infty}^{\infty} \exp\{-i\omega y\} \Psi'_X \, (\Psi(\omega)) \, \Psi'(\omega) d\omega. \tag{51}$$

Now, substituting Eq. (50) into Eq. (51) and rearranging terms gives

$$\lambda_Y(y) = \int_0^{\infty} \left\{ \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp\{-i\omega y\}}{iy} \exp\{-\left(\Psi(\omega)x\right)\} \left(\Psi'(\omega)x\right) d\omega \right\} \lambda_X(x) dx,$$

and, after integration by parts of the integrand, we arrive at

$$\lambda_Y(y) = \int_0^{\infty} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-i\omega y\} \exp\{-\Psi(\omega)x\} d\omega \right\} \lambda_X(x) dx. \tag{52}$$

Finally, using the inverse Fourier transform (34), Eq. (52) yields

$$\lambda_Y(y) = \int_0^{\infty} f(x; y) \lambda_X(x) dx. \quad \blacksquare$$

### A.3. Equation (46)

*Proof.* Since $X$ is $\alpha$-selfsimilar with amplitude $2^\alpha$, and since $Y$ is $2\alpha$-selfsimilar with unit amplitude we have (using the selfsimilar examples of subsection 2.4):

$$g_X(x) := x\lambda_X(x) = \frac{2^\alpha \alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{x^\alpha}, \tag{53}$$

and

$$g_Y(y) := y\lambda_Y(y) = \frac{\alpha}{I_{2\alpha}} \cdot \frac{1}{y^{2\alpha}}, \tag{54}$$

where $I_{2\alpha} = \int_0^\infty \sin\{u\} u^{-2\alpha} du$.

Substituting (53)–(54) into Eq. (44) gives

$$\frac{1}{I_{2\alpha}} = \frac{2^\alpha}{\Gamma(1-\alpha)} \mathbf{E}\left[|Z|^{2\alpha}\right],$$

where $Z$ is normalized Gaussian. However, since $Z$ is normalized Gaussian we have $\mathbf{E}\left[|Z|^{2\alpha}\right] = 2^\alpha \Gamma(\frac{1}{2}+\alpha)/\sqrt{\pi}$, and hence

$$I_{2\alpha} = \frac{\sqrt{\pi}}{4^\alpha} \frac{\Gamma(1-\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\right)}. \quad \blacksquare$$

### A.4. Equation (47)

*Proof.* Since $X$ is $\alpha$-selfsimilar and $Y$ is $\alpha\beta$-selfsimilar – both with unit amplitude – we have (using the selfsimilar examples of subsection 2.4):

$$g_X(x) := x\lambda_X(x) = \frac{\alpha}{\Gamma(1-\alpha)} \cdot \frac{1}{x^\alpha}, \tag{55}$$

and

$$g_Y(y) := y\lambda_Y(y) = \frac{\alpha\beta}{2I_{\alpha\beta}} \cdot \frac{1}{y^{\alpha\beta}}, \tag{56}$$

where $I_{\alpha\beta} = \int_0^\infty \sin\{u\} u^{-\alpha\beta} du$.

Substituting Eqs. (55)–(56) into Eq. (43) gives

$$\frac{1}{I_{\alpha\beta}} = \frac{1}{\Gamma(1-\alpha)}\mathbf{E}\left[|L(1)|^{\alpha\beta}\right].$$

Hence, setting $p=\alpha\beta$ and using (46) we obtain that

$$\mathbf{E}\left[|L(1)|^{p}\right] = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{1+p}{2}\right)\frac{\Gamma(1-p/\beta)}{\Gamma(1-p/2)}. \tag{57}$$

Finally, since $L(t)\stackrel{d}{=}t^{1/\beta}L(1)$, Eq. (57) yields

$$\mathbf{E}\left[|L(t)|^{p}\right] = \frac{2^p}{\sqrt{\pi}}\Gamma\left(\frac{1+p}{2}\right)\frac{\Gamma(1-p/\beta)}{\Gamma(1-p/2)}\cdot t^{p/\beta}. \quad \blacksquare$$

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